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Canonical form for H-symplectic matrices

G.J. Groenewald, D.B. Janse van Rensburg and A.C.M. Ran

Dedicated to Rien Kaashoek on the occasion of his eightieth birthday

Abstract. In this paper we consider pairs of matrices (A, H), with A and H either both real or both complex, H is invertible and skew-symmetric and A is H-symplectic, that is, $A^T HA = H$. A canonical form for such pairs is derived under the transformations $(A, H) \to (S^{-1}AS, S^T HS)$ for invertible matrices S. In the canonical form for the pair, the matrix A is brought in standard (real or complex) Jordan normal form, in contrast to existing canonical forms.

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1. Introduction

In this paper we shall consider pairs of matrices (A, H), where A and H are either real or complex matrices and A is H-symplectic. Recall, when $H = -H^T$ is invertible, a matrix A is called H-symplectic when

$$A^T HA = H.$$

Obviously, when S is an invertible matrix, then $S^{-1}AS$ is $S^T HS$-symplectic. Under these transformations one might ask: what is the canonical form for the pair (A, H)? Such canonical forms already exist in the literature, see for instance [14, 15], and [4, 9, 16, 17, 18] for several slightly different versions. The canonical forms available in the literature keep H in as simple a form as possible, and simultaneously bring A into a form from which the Jordan canonical form of A may be read off more or less easily, with blocks that are at best of the form $J_n(\lambda) \oplus J_n(\frac{1}{\lambda})^{-T}$, where the superscript $-T$ indicates the transpose of the inverse. (As usual, $J_n(\lambda)$ denotes the $n \times n$ upper triangular Jordan block with eigenvalue λ.) In some cases blocks in the canonical form are much more complicated. It is our goal to present here a canonical form where A is completely in (real) Jordan form.

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In our previous paper [8] we considered matrices A which were unitary in an indefinite inner product given by a symmetric (or Hermitian) matrix H. Canonical forms for unitary matrices in indefinite inner product spaces have been the subject of many papers, we mention here the books [6, 7] where the canonical forms were deduced from corresponding canonical forms for selfadjoint matrices in an indefinite inner product space, and the papers [10, 19, 20, 21, 22, 23], see also [1]. General theory for operators and matrices in indefinite inner product spaces may be found in the books [2, 3, 6, 7, 11]. We shall make use of results from [14, 15] where unitary and symplectic matrices are studied from the point of view of normal matrices in an indefinite inner product space, and where also canonical forms have been given. Closest to our development in [8] is the paper [13], although a complete canonical form is not deduced there. Our point of view is that we wish to bring the matrix A in (real) Jordan canonical form, and see what this implies for the matrix H representing the bilinear or sesquilinear form. The start of our considerations was the simple form for expansive matrices in an indefinite inner product, developed in [12] and [5].

We consider both the complex case, as well as the real case, where all matrices involved are assumed to be real. In fact, there are three cases to be considered:

1. A and H are complex matrices, with $H = -H^*$ invertible and $A^*HA = H$, considered as matrices acting in the space $\mathbb{C}^n$ equipped with the standard sesquilinear form $\langle x, y\rangle = \sum_{i=1}^n x_i\overline{y_i}$,
2. A and H are complex matrices, with $H = -H^T$ invertible and $A^THA = H$, considered as matrices acting in the space $\mathbb{C}^n$ equipped with the standard bilinear form $\langle x, y\rangle = \sum_{i=1}^n x_iy_i$,
3. A and H are real matrices,with $H = -H^T$ invertible and $A^THA = H$, considered as matrices acting in the space $\mathbb{R}^n$ equipped with the standard bilinear form.

The first case is easy: put $H_1 = iH$, then $H_1 = H_1^*$ is invertible and $A^*H_1A = H_1$. Hence, A is H_1-unitary, and a canonical form can be deduced from canonical forms for unitary matrices in indefinite inner product spaces, such as given in, e.g., our recent paper [8]. The resulting theorem is presented in the final section of this paper.

The focus of this paper will be on the two remaining cases above. In the third case the matrix A will be called H-symplectic; this is the classical case. The second case is less well-studied. In that case, for lack of a better term, we shall call the matrix A also H-symplectic, it will be clear from the context whether we work in $\mathbb{C}^n$ or $\mathbb{R}^n$. It should be stressed that in both cases the space is equipped with the standard bilinear form (so, in particular, in the complex case *not* the standard sesquilinear form). A canonical form for the second case seems to be less well-known, and probably appeared for the first time in [14].

It is our aim to derive a canonical form with A in standard (real or complex) Jordan canonical form (see [24]). This would be analogous to the canonical form we recently derived for H-unitary matrices in [8]. Of course, starting from any canonical form where A is not exactly in Jordan canonical form, we can transform the pair (A, H) via an appropriate basis transformation S to the pair $(S^{-1}AS, S^THS)$ with $S^{-1}AS$ in Jordan canonical form. This will have the desired effect, although it may not directly produce the same form for S^THS that is achieved in our main results. Indeed, this is caused by the fact that there are many invertible matrices S such that, when A is in Jordan canonical form, also $S^{-1}AS$ is in Jordan canonical form. We shall make use of the freedom this provides in our proofs. The authors thank the anonymous referee for pointing out that the main results (Theorem 2.6 and Theorem 2.11) of this paper may be derived in this manner from results in paper [16] (albeit with a considerable amount of work for some of the cases). However, we have chosen to take a more direct approach here, and develop the desired canonical form from scratch. To further motivate our choice to keep A in Jordan canonical form, consider the problem of finding a function $f(A)$ of the matrix A, which is greatly facilitated by having A in Jordan normal form.

We can once again use the results on the indecomposable blocks ([14, 15]) to limit the number of cases we have to consider. In particular, Theorem 8.5 in [14] gives a canonical form, but also tells us that the indecomposable blocks in the complex case are of three types and is given in the following proposition, where U in [14] is replaced by our A, and Q in [14] is our S.

Proposition 1.1. *Let $H = -H^T$ be invertible and let A be H-symplectic. Then there is an invertible matrix S such that*

$$S^{-1}AS = \oplus_{j=1}^{k} A_j, \qquad S^THS = \oplus_{j=1}^{k} H_j, \tag{1.1}$$

where in each pair (A_j, H_j) the matrix $H_j = -H_j^T$ is invertible, and A_j is H_j-symplectic, and each pair is of one of the following indecomposable forms:

(i) (*complex eigenvalues*)

$$A_j = J_{n_j}(\lambda) \oplus J_{n_j}(\frac{1}{\lambda}) \text{ with } \operatorname{Re}\lambda > \operatorname{Re}\frac{1}{\lambda} \text{ or } \operatorname{Im}\lambda > \operatorname{Im}\frac{1}{\lambda} \text{ if}$$
$$\operatorname{Re}\lambda = \operatorname{Re}\frac{1}{\lambda}, \tag{1.2}$$
$$H_j = \begin{bmatrix} 0 & H_{12} \\ -H_{12}^T & 0 \end{bmatrix};$$

(ii) ± 1, *even partial multiplicity,*

$$A_j = J_{n_j}(\pm 1), \text{ with } n_j \text{ even}, \; H_j = -H_j^T; \tag{1.3}$$

(iii) ± 1, *odd partial multiplicities,*

$$A_j = J_{n_j}(\pm 1) \oplus J_{n_j}(\pm 1) \text{ with } n_j \text{ odd}, \; H_j = \begin{bmatrix} 0 & H_{12} \\ -H_{12}^T & 0 \end{bmatrix}. \tag{1.4}$$

The matrices H_j, and in particular the form of the matrices H_{12} in (1.2) and (1.4) may be further reduced to a canonical form as is described in the main results of this paper.

In [15], Theorem 5.5, the canonical form and the indecomposable blocks for the real case are described, in that case the first class of blocks for the complex case has to be subdivided into three classes: when λ is real, when λ is unimodular but not ± 1, and when λ is non-real and non-unimodular (in the latter case actually there is a quadruple of eigenvalues involved).

In particular, note that odd sized blocks with eigenvalue one or minus one come in pairs. This was proved in e.g. [14], see in particular Proposition 3.4 there and its proof, and also in [1], Proposition 3.1.

As a consequence of this, all one needs to do to arrive at a canonical form for the pair (A, H) is to derive canonical forms for each of these indecomposable blocks.

2. Main Results

In this section we will present the main results of this article. In the first subsection is the main result for the complex case and in the second subsection the main result for the real case. Each subsection makes use of a number of definitions which will be presented first. Most of these definitions have their origin in the canonical form for H-unitary matrices described in [8].

2.1. Complex case

We start by giving the definitions needed for the main theorem in the complex case.

Definition 2.1. For odd $n > 1$ the $\frac{n+1}{2} \times \frac{n-1}{2}$ matrix $P_n = \left[p_{ij}\right]_{i=1,j=1}^{\frac{n+1}{2},\frac{n-1}{2}}$ is defined as follows:

$$\begin{aligned}
&p_{i\,j} = 0 && \text{when } i + j \leq \frac{n-1}{2},\\
&p_{i\,\frac{n-1}{2}-i+1} = (-1)^{\frac{n-1}{2}-i+1} && \text{for } i = 1, \dots, \frac{n+1}{2} - 1,\\
&p_{\frac{n+1}{2}\,j} = (-1)^j \cdot \frac{1}{2} && \text{for } j = 1, \dots, \frac{n-1}{2},
\end{aligned}$$

and all other entries are defined by $p_{i\ j+1} = -(p_{i\,j} + p_{i+1\,j})$.

The numbers p_{ij} have the following explicit formula, with the understanding that $\binom{j}{k} = 0$ whenever $k < 0$ or $j < k$:

$$p_{ij} = \frac{(-1)^j}{2} \left(\binom{j+1}{\frac{n+1}{2} - i} - \binom{j-1}{\frac{n+1}{2} - i - 2} \right).$$

Indeed, it can be easily checked that these numbers satisfy the recursion and the initial values given in Definition 2.1.

To get a feeling for how such a matrix looks like, we give P_{11} below:

$$P_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -\frac{9}{2} \\ 0 & 0 & -1 & \frac{7}{2} & -\frac{16}{2} \\ 0 & 1 & -\frac{5}{2} & \frac{9}{2} & -\frac{14}{2} \\ -1 & \frac{3}{2} & -\frac{4}{2} & \frac{5}{2} & -\frac{6}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Also, P_9 is the submatrix of P_{11} formed by deleting the last column and first row.

Observe that the recursion for the entries of P_n actually holds for all its entries, provided the first column and last row are given, or the last column and first row. Also note that the recursion is the same as the one for Pascal's triangle, modulo a minus sign. Take note that the entries are not the numbers in the Pascal triangle because the starting values are different: if we consider the entries in the first column and last row of P_n as the starting values, then the nonzero starting numbers are $1, \frac{1}{2}$ rather than $1, 1$ as would be the case for the Pascal triangle.

Definition 2.2. We also introduce for odd n the $\frac{n+1}{2} \times \frac{n+1}{2}$ matrix Z_n which has zeros everywhere, except in the $(\frac{n+1}{2}, \frac{n+1}{2})$-entry, where it has a 1. For instance, Z_5 is given by

$$Z_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We shall also make use of the matrices $Z_n \otimes I_2$, which is the $(n+1) \times (n+1)$ matrix which has zeros everywhere except in the two by two lower right block where it has I_2, and $Z_n \otimes K_1$, where $K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, i.e., the $(n+1) \times (n+1)$ matrix which has zeros everywhere except in the two by two lower right block where it has K_1.

Definition 2.3. Next we introduce for even n the $\frac{n}{2} \times \frac{n}{2}$ matrix $Q_n = \left[q_{ij}\right]_{i=1,\, j=1}^{\frac{n}{2},\, \frac{n}{2}}$ as follows:

$$\begin{aligned} q_{i\,j} &= 0 && \text{when } i + j \leq \frac{n}{2}, \\ q_{i\ \frac{n}{2}-i+1} &= (-1)^{\frac{n}{2}-i} && \text{for } i = 1, \ldots, \frac{n}{2}, \\ q_{\frac{n}{2}\,j} &= (-1)^{j-1} && \text{for } j = 1, \ldots, \frac{n}{2}, \end{aligned}$$

and all other entries are defined by $q_{i\ j+1} = -(q_{i\,j} + q_{i+1\,j})$.

Again, we give an example: Q_{10} is given by

$$Q_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & -3 & 6 \\ 0 & -1 & 2 & -3 & 4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

Also, Q_8 is formed from this by deleting the first row and last column. Note that the numbers involved, apart from a minus sign, are exactly the numbers from Pascal's triangle, so in this case we can even give a precise formula: when $i + j \geq \frac{n}{2} + 1$ we have

$$q_{ij} = (-1)^{j-1} \binom{j-1}{\frac{n}{2} - i}.$$

For $\lambda \in \mathbb{C} \setminus \{-1, 0, 1\}$ we define the following:

Definition 2.4. Let $n > 1$ be an odd integer, then the $\frac{n+1}{2} \times \frac{n-1}{2}$ matrix $P_n(\lambda)$ is defined as follows:

$$P_n(\lambda) = \left[p_{ij} \lambda^{\frac{n+1}{2} + j - i} \right]_{i=1,\ j=1}^{\frac{n+1}{2},\ \frac{n-1}{2}} \tag{2.1}$$

where p_{ij} are the entries of the matrix P_n introduced above.

For example, $P_5(\lambda)$ is the 3×2 matrix given by

$$P_5(\lambda) = \begin{bmatrix} 0 & \lambda^4 \\ -\lambda^2 & \frac{3}{2}\lambda^3 \\ -\frac{1}{2}\lambda & \frac{1}{2}\lambda^2 \end{bmatrix}.$$

Definition 2.5. Let $n > 1$ be an even integer, then the $\frac{n}{2} \times \frac{n}{2}$ matrix $Q_n(\lambda)$ is defined as follows:

$$Q_n(\lambda) = \left[q_{ij} \lambda^{\frac{n}{2} + j - i - 1} \right]_{i=1,\ j=1}^{\frac{n}{2},\ \frac{n}{2}} \tag{2.2}$$

where q_{ij} are the entries of the matrix Q_n introduced earlier.

With these definitions in place we state now the main theorem for the complex case. The reader should realize that in P_n, Q_n, Z_n the subscript n does not indicate that these are $n \times n$ matrices, but that the dimensions of these matrices depend on n as indicated in the previous definitions.

Theorem 2.6. *Let A be H-symplectic, with both A and H complex. Then the pair (A, H) can be decomposed as follows. There is an invertible matrix S such that*

$$S^{-1} A S = \bigoplus_{l=1}^{p} A_l, \quad S^T H S = \bigoplus_{l=1}^{p} H_l,$$

where each pair (A_l, H_l) satisfies one of the following conditions for some n depending on l;

(i) $\sigma(A_l) = \{1\}$ *and the pair (A_l, H_l) has one of the following two forms:*

Case 1: $\left(J_n(1)\oplus J_n(1), \begin{bmatrix} 0 & 0 & Z_n & P_n \\ 0 & 0 & P_n^T & 0 \\ -Z_n^T & -P_n & 0 & 0 \\ -P_n^T & 0 & 0 & 0 \end{bmatrix}\right)$ *with n odd.*

Case 2: $\left(J_n(1), \begin{bmatrix} 0 & Q_n \\ -Q_n^T & 0 \end{bmatrix}\right)$ *with n even.*

(ii) $\sigma(A_l)=\{-1\}$ *and the pair (A_l,H_l) has one of the following two forms:*

Case 1: $\left(J_n(-1)\oplus J_n(-1), \begin{bmatrix} 0 & 0 & Z_n & P_n(-1) \\ 0 & 0 & P_n^T(-1) & 0 \\ -Z_n^T & -P_n(-1) & 0 & 0 \\ -P_n^T(-1) & 0 & 0 & 0 \end{bmatrix}\right)$ *with n odd.*

Case 2: $\left(J_n(-1), \begin{bmatrix} 0 & Q_n(-1) \\ -Q_n^T(-1) & 0 \end{bmatrix}\right)$ *with n even.*

(iii) $\sigma(A_l)=\{\lambda,\frac{1}{\lambda}\}$ *with $\lambda\in\mathbb{C}\backslash\{-1,0,1\}$ and the pair (A_l,H_l) is of the form* $\left(J_n(\lambda)\oplus J_n(\frac{1}{\lambda}), \begin{bmatrix} 0 & H_{12} \\ -H_{12}^T & 0 \end{bmatrix}\right)$, *where H_{12} is of one of the following two forms, depending on whether n is odd or even:*

Case 1: *n is odd:* $H_{12}=\begin{bmatrix} Z_n & P_n(\lambda) \\ P_n(\frac{1}{\lambda})^T & 0 \end{bmatrix}$.

Case 2: *n is even:* $H_{12}=\begin{bmatrix} 0 & Q_n(\lambda) \\ -\frac{1}{\lambda^2}Q_n(\frac{1}{\lambda})^T & 0 \end{bmatrix}$.

2.2. Real case

First we present a number of definitions needed for the main theorem of the real case. Take note that analogues of some of the definitions presented earlier in the complex case are also needed in the real case.

Let $\gamma=\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ with $\beta\neq 0$ and $\alpha^2+\beta^2=1$. As usual $J_n(\gamma)$ denotes the matrix

$$J_n(\gamma)=\begin{bmatrix} \gamma & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & \gamma \end{bmatrix}$$

of size $2n\times 2n$. We also define the following:

Definition 2.7. For n odd the $\frac{n+1}{2}\times\frac{n-1}{2}$ block matrix $\tilde{P}_n(\gamma)$ with two by two matrix blocks is defined as:

$$\tilde{P}_n(\gamma)=\left[p_{ij}H_0(\gamma^T)^{\frac{n+1}{2}+j-i}\right]_{i=1,\ \ j=1}^{\frac{n+1}{2},\ \frac{n-1}{2}},$$

where p_{ij} are the entries of the matrix P_n as in Definition 2.1, and $H_0=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

As an example,

$$\tilde{P}_5(\gamma) = \begin{bmatrix} 0 & H_0(\gamma^T)^4 \\ -H_0(\gamma^T)^2 & \frac{3}{2}H_0(\gamma^T)^3 \\ -\frac{1}{2}H_0\gamma^T & \frac{1}{2}H_0(\gamma^T)^2 \end{bmatrix}$$

Definition 2.8. For even n the $\frac{n}{2} \times \frac{n}{2}$ block matrix $\tilde{Q}_n(\gamma)$ with two by two matrix blocks is defined as:

$$\tilde{Q}_n(\gamma) = \left[q_{ij}(\gamma^T)^{\frac{n}{2}+j-i}\right]_{i=1,\ \ j=1}^{\frac{n}{2},\ \frac{n}{2}},$$

where q_{ij} are the entries of the matrix Q_n as in Definition 2.3.

Also, for $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, with $\alpha^2 + \beta^2 \neq 1$, we define

Definition 2.9. Let $n > 1$ be an odd integer, then the $\frac{n+1}{2} \times \frac{n-1}{2}$ block matrix $P_n(\gamma)$ with two by two matrix blocks is defined as follows:

$$P_n(\gamma) = \left[p_{ij}K_1\gamma^{\frac{n+1}{2}+j-i}\right]_{i=1,\ j=1}^{\frac{n+1}{2},\ \frac{n-1}{2}} \tag{2.3}$$

where p_{ij} are the entries of the matrix P_n introduced earlier, and $K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

For example, $P_5(\gamma)$ is the 3×2 matrix given by

$$P_5(\gamma) = \begin{bmatrix} 0 & K_1\gamma^4 \\ -K_1\gamma^2 & \frac{3}{2}K_1\gamma^3 \\ -\frac{1}{2}K_1\gamma & \frac{1}{2}K_1\gamma^2 \end{bmatrix}.$$

Definition 2.10. Let $n > 1$ be an even integer, then the $\frac{n}{2} \times \frac{n}{2}$ block matrix $Q_n(\gamma)$ is defined as follows:

$$Q_n(\gamma) = \left[q_{ij}K_1\gamma^{\frac{n}{2}+j-i-1}\right]_{i=1,\ j=1}^{\frac{n}{2},\ \frac{n}{2}} \tag{2.4}$$

where q_{ij} are the entries of the matrix Q_n introduced earlier.

With these definitions in place, we state the main theorem in the real case.

Theorem 2.11. *Let A be H-symplectic, with both A and H real. Then the pair (A, H) can be decomposed as follows. There is an invertible real matrix S such that*

$$S^{-1}AS = \bigoplus_{l=1}^{p} A_l, \quad S^THS = \bigoplus_{l=1}^{p} H_l,$$

where each pair (A_l, H_l) satisfies one of the following conditions for some n depending on l;

(i) $\sigma(A_l) = \{1\}$ *and the pair (A_l, H_l) has one of the following two forms:*

Case 1: $\left(J_n(1) \oplus J_n(1), \begin{bmatrix} 0 & 0 & Z_n & P_n \\ 0 & 0 & P_n^T & 0 \\ -Z_n^T & -P_n & 0 & 0 \\ -P_n^T & 0 & 0 & 0 \end{bmatrix}\right)$ *with n odd.*

Case 2: $\left(J_n(1), \varepsilon \begin{bmatrix} 0 & Q_n \\ -Q_n^T & 0 \end{bmatrix}\right)$ *with n even, and $\varepsilon = \pm 1$.*

(ii) $\sigma(A_l) = \{-1\}$ *and the pair (A_l, H_l) has one of the following two forms:*

Case 1: $\left(J_n(-1) \oplus J_n(-1), \begin{bmatrix} 0 & 0 & Z_n & P_n(-1) \\ 0 & 0 & P_n^T(-1) & 0 \\ -Z_n^T & -P_n(-1) & 0 & 0 \\ -P_n^T(-1) & 0 & 0 & 0 \end{bmatrix}\right)$ *with n odd.*

Case 2: $\left(J_n(-1), \varepsilon \begin{bmatrix} 0 & Q_n(-1) \\ -Q_n^T(-1) & 0 \end{bmatrix}\right)$ *with n even, and $\varepsilon = \pm 1$.*

(iii) $\sigma(A_l) = \{\lambda, \frac{1}{\lambda}\}$ *with $\lambda \in \mathbb{R}\backslash\{-1, 0, 1\}$ and the pair (A_l, H_l) is of the form $\left(J_n(\lambda) \oplus J_n(\frac{1}{\lambda}), \begin{bmatrix} 0 & H_{12} \\ -H_{12}^T & 0 \end{bmatrix}\right)$, where H_{12} is of one of the following two forms, depending on whether n is odd or even:*

Case 1: *n is odd:* $H_{12} = \begin{bmatrix} Z_n & P_n(\lambda) \\ P_n(\frac{1}{\lambda})^T & 0 \end{bmatrix}$.

Case 2: *n is even:* $H_{12} = \begin{bmatrix} 0 & Q_n(\lambda) \\ -\frac{1}{\lambda^2} Q_n(\frac{1}{\lambda})^T & 0 \end{bmatrix}$.

(iv) $\sigma(A_l) = \{\alpha \pm i\beta\}$ *with $\alpha^2 + \beta^2 = 1$ and $\beta \neq 0$, and the pair (A_l, H_l) has one of the following two forms with $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$:*

Case 1: $\left(J_n(\gamma), \varepsilon \begin{bmatrix} Z_n \otimes H_0 & \tilde{P}_n(\gamma) \\ -\tilde{P}_n(\gamma)^T & 0 \end{bmatrix}\right)$ *with n odd, and $\varepsilon = \pm 1$.*

Case 2: $\left(J_n(\gamma), \varepsilon \begin{bmatrix} 0 & \tilde{Q}_n(\gamma) \\ -\tilde{Q}_n(\gamma)^T & 0 \end{bmatrix}\right)$ *with n even, and $\varepsilon = \pm 1$.*

(v) $\sigma(A_l) = \{\alpha \pm i\beta, (\alpha \pm i\beta)^{-1}\}$ *with $\alpha^2 + \beta^2 \neq 1$ and $\beta \neq 0$, and the pair (A_l, H_l) is of the form $\left(J_n(\gamma) \oplus J_n(\gamma^{-1}), \begin{bmatrix} 0 & H_{12} \\ -H_{12}^T & 0 \end{bmatrix}\right)$, where H_{12} is of one of the following two forms, depending on whether n is odd or even:*

Case 1: *n is odd:* $H_{12} = \begin{bmatrix} Z_n \otimes K_1 & P_n(\gamma) \\ P_n(\gamma^{-1})^T & 0 \end{bmatrix}$.

Case 2: *n is even:* $H_{12} = \begin{bmatrix} 0 & Q_n(\gamma) \\ -(I_n \otimes -\gamma^{-2}) Q_n(\gamma^{-1})^T & 0 \end{bmatrix}$.

Note that the columns of the matrix S in the theorem form a special real Jordan basis for A. The theorem should be compared with the canonical form obtained in [15], in particular with Theorem 5.5 there.

The proofs of Theorems 2.6 and 2.11 will be given in the sections to follow. In subsection 3.1 part (i) Case 1 and 2 is proved with $\lambda = 1$ for the complex case. Part (ii) follows similarly. Part (iii) needs no proof as this is equivalent to what has been done in the unitary case.

The real case is proved in Section 4. Parts (i) and (ii) are the same for the real and complex case, as is part (iii). Part (v) is the same as in the unitary case, so needs no extra proof. Only part (iv) needs a new proof here, and this is presented in Section 4 in detail.

The main theorem for the complex H-symplectic case is given in the final section of this paper without proof, as the proof of this theorem once again follows directly from the unitary case, see [8].

3. The complex case

We prove the main theorem, Theorem 2.6, of the complex case in the subsections to follow.

3.1. Eigenvalue one

We begin by proving Case 1 (with n odd) of Part (i) of Theorem 2.6.

Proof. We may assume, based on Proposition 1.1, that in this case

$$A = J_n(1) \oplus J_n(1), \qquad H = \begin{bmatrix} 0 & H^{(0)} \\ -(H^{(0)})^T & 0 \end{bmatrix}$$

for an invertible $n \times n$ matrix $H^{(0)}$. By Section 2 in [5], (see also [12]), we have that $H^{(0)}_{i\,j} = 0$ for $i + j \leq n$. Moreover, from $A^T HA = H$ we have that

$$J_n(1)^T H^{(0)} J_n(1) = H^{(0)},$$

and so

$$H^{(0)}_{i\,j} + H^{(0)}_{i\,j+1} + H^{(0)}_{i+1\,j} = 0, \quad \text{for } i > 1 \text{ and } j > 1. \tag{3.1}$$

Consider $S = \hat{S} \oplus \tilde{S}$, where $\hat{S}$ and $\tilde{S}$ are $n \times n$ upper triangular Toeplitz matrices, in particular, the upper triangular Toeplitz matrix $\hat{S}$ with first row $h_1, \cdots, h_n$ will be denoted by $\operatorname{toep}(h_1, \cdots, h_n)$. Then $S^{-1}AS = A$, and

$$S^T HS = \begin{bmatrix} 0 & \hat{S}^T H^{(0)} \tilde{S} \\ -\tilde{S}^T (H^{(0)})^T \hat{S} & 0 \end{bmatrix},$$

and note that S is chosen such that (3.1) holds, with H replaced by $H' = S^T HS$ and $H'_{ij} = 0$ for $i + j \leq 0$. Indeed, we take $\tilde{S} = I$ and will show that $\hat{S}$ can be chosen so that $\hat{S}^T H^{(0)}$ has the form given in Part (i) Case 1 of both Theorems 2.6 and 2.11, that is,

$$\tilde{H} = \hat{S}^T H^{(0)} = \begin{bmatrix} Z_n & P_n \\ P_n^T & 0 \end{bmatrix}.$$

In order to achieve this, it suffices, in view of (3.1) to show that $\hat{S}$ can be chosen such that

$$\tilde{H}_{\frac{n+1}{2}\ \frac{n+1}{2}} = 1, \quad \tilde{H}_{\frac{n+1}{2}\ \frac{n+1}{2}+1} = -\frac{1}{2},$$
$$\tilde{H}_{\frac{n+1}{2}+j\ \frac{n+1}{2}+j} = 0, \text{ for } j = 1, \ldots, \frac{n-1}{2},$$
$$\tilde{H}_{\frac{n+1}{2}+j\ \frac{n+1}{2}+j+1} = 0, \text{ for } j = 1, \ldots, \frac{n-1}{2} - 1.$$

Let $\hat{S} = \operatorname{toep}(h_1, \cdots, h_n)$, and compute $\tilde{H}_{i\,j} = (\hat{S}^T H^{(0)})_{i\,j}$:

$$\tilde{H}_{i\,j} = \begin{cases} 0 & \text{if } i + j \leq n, \\ h_1 H^{(0)}_{i\,j} + h_2 H^{(0)}_{i-1\,j} + \cdots + h_{i+j-n} H^{(0)}_{n+1-j\,j} & \text{if } i + j > n. \end{cases} \tag{3.2}$$

Put $S_1 = h_1 I_n$, and $H^{(1)} = S_1^T H^{(0)}$. For $i+j = n+1$ with $i = \frac{n+1}{2}$ and $j = \frac{n+1}{2}$, we have

$$H^{(1)}_{\frac{n+1}{2}\ \frac{n+1}{2}} = h_1 H^{(0)}_{\frac{n+1}{2}\ \frac{n+1}{2}}.$$

Because of the invertibility of $H^{(0)}$ we can take h_1 so that $\frac{1}{h_1} = H^{(0)}_{\frac{n+1}{2}\ \frac{n+1}{2}}$, and hence $H^{(1)}_{\frac{n+1}{2}\ \frac{n+1}{2}} = 1$. Formula (3.1) with $H^{(0)}$ replaced by $H^{(1)}$, ensures that all anti-diagonal entries of $H^{(1)}$ with $i + j = n + 1$ alternates between $+1$ and -1.

Now let $S_2 = \operatorname{toep}(1, h_2, 0, \cdots, 0)$, and let $H^{(2)} = S_2^T H^{(1)}$. Then for $i + j > n$ we have from (3.2) that

$$H^{(2)}_{i\,j} = H^{(1)}_{i\,j} + h_2 H^{(1)}_{i-1\,j}. \tag{3.3}$$

In particular, for $i + j = n + 1$ we have

$$H^{(2)}_{i\ n+1-i} = H^{(1)}_{i\ n+1-i} + h_2 H^{(1)}_{i-1\ n+1-i} = H^{(1)}_{i\ n+1-i},$$

and so $H^{(2)}_{i\,j} = H^{(1)}_{i\,j}$ for $i+j \leq n+1$. Furthermore, for $i+j = n+2$ we have from (3.3) that

$$H^{(2)}_{i\ n+2-i} = H^{(1)}_{i\ n+2-i} + h_2 H^{(1)}_{i-1\ n+2-i}.$$

We have $H^{(1)}_{i-1\ n+2-i} = \pm 1$. In particular, for $i = \frac{n+1}{2}$ ($j = \frac{n+1}{2} + 1$), we have

$$H^{(2)}_{\frac{n+1}{2}\ \frac{n+1}{2}+1} = H^{(1)}_{\frac{n+1}{2}\ \frac{n+1}{2}+1} + h_2 H^{(1)}_{\frac{n+1}{2}-1\ \frac{n+1}{2}+1} = H^{(1)}_{\frac{n+1}{2}\ \frac{n+1}{2}+1} - h_2,$$

since $H^{(1)}_{\frac{n+1}{2}-1\ \frac{n+1}{2}+1} = -1$. Now we can choose h_2 such that $H^{(2)}_{\frac{n+1}{2}\ \frac{n+1}{2}+1} = -\frac{1}{2}$. A repeated application of (3.1) determines all entries for which $i + j = n + 2$.

Next, put $S_3 = \operatorname{toep}(1, 0, h_3, 0, \cdots, 0)$ and $H^{(3)} = S_3^T H^{(2)}$. By (3.2) we have for $i + j > n$ that

$$H^{(3)}_{i\,j} = H^{(2)}_{i\,j} + h_3 H^{(2)}_{i-2\,j}.$$

If $i+j \le n+2$ this gives $H^{(3)}_{i\,j} = H^{(2)}_{i\,j}$, since $i+j-2 \le n$, so that $H^{(2)}_{i-2\,j} = 0$. For $i+j = n+3$ we obtain from the identity above that

$$H^{(3)}_{i\ n+3-i} = H^{(2)}_{i\ n+3-i} + h_3 H^{(2)}_{i-2\ n+3-i}.$$

For $i = j = \frac{n+1}{2} + 1$, we have

$$H^{(3)}_{\frac{n+1}{2}+1\ \frac{n+1}{2}+1} = H^{(2)}_{\frac{n+1}{2}+1\ \frac{n+1}{2}+1} + h_3 H^{(2)}_{\frac{n+1}{2}-1\ \frac{n+1}{2}+1}.$$

Since $H^{(2)}_{\frac{n+1}{2}-1\ \frac{n+1}{2}+1} = -1$ we can take $h_3 = H^{(2)}_{\frac{n+1}{2}+1\ \frac{n+1}{2}+1}$ to obtain $H^{(3)}_{\frac{n+1}{2}+1\ \frac{n+1}{2}+1} = 0$. Similar as before, this can be used to determine all entries $H^{(3)}_{i\,j}$ for which $i+j = n+3$ by using (3.1).

Now put $S_4 = \mathrm{toep}(1, 0, 0, h_4, 0, \ldots, 0)$ and $H^{(4)} = S_4^T H^{(3)}$. From (3.2) we have for $i+j > n$ that

$$H^{(4)}_{i\,j} = H^{(3)}_{i\,j} + h_4 H^{(3)}_{i-3\,j}.$$

As before $H^{(4)}_{i\,j} = H^{(3)}_{i\,j}$ for all $i+j \le n+3$. For $i+j = n+4$ we have from (3.2) that

$$H^{(4)}_{i\ n+4-i} = H^{(3)}_{i\ n+4-i} + h_4 H^{(3)}_{i-3\ n+4-i}.$$

If we take $i = \frac{n+1}{2} + 1$, then $j = \frac{n+1}{2} + 2$ and we have

$$H^{(4)}_{\frac{n+1}{2}+1\ \ \frac{n+1}{2}+2} = H^{(3)}_{\frac{n+1}{2}+1\ \ \frac{n+1}{2}+2} + h_4 H^{(3)}_{\frac{n+1}{2}-2\ \ \frac{n+1}{2}+2} = H^{(3)}_{\frac{n+1}{2}+1\ \ \frac{n+1}{2}+2} + h_4,$$

since $H^{(3)}_{\frac{n+1}{2}-2\ \ \frac{n+1}{2}+2} = 1$. So, we can choose h_4 such that $H^{(4)}_{\frac{n+1}{2}+1\ \ \frac{n+1}{2}+2} = 0$. By repeated application of updates of (3.1), all entries of $H^{(4)}_{i\,j}$ for which $i+j = n+4$ can now be computed. Now we can continue by induction to finish the proof, where $\hat{S} = S_{\frac{n+1}{2}} \cdots S_2\, S_1$. □

Proof of Theorem 2.6 *Part* (i) *Case* 2 *with n even.* We may assume that $A = J_n(1)$ with respect to the Jordan basis $x_1, \ldots, x_n$. Denote $H = [H_{ij}]_{i,j=1}^n$ (so $H_{ij} = \langle Hx_j, x_i\rangle$). Analogously to what has been done in [5] and [12], we already know the following:

$$H_{ij} = 0 \quad \text{when} \quad i+j \le n,$$

and

$$H_{ij} + H_{i\,j+1} + H_{i+1\,j} = 0 \quad \text{for } i > 1 \text{ and } j > 1. \tag{3.4}$$

Furthermore, we have from the fact that H is skew-symmetric that all entries on the main diagonal are zero and we have that $H_{ij} = -H_{ji}$.

Let us denote for convenience $H_{\frac{n}{2}\ \frac{n}{2}+1} := c$, where c is complex and $c \ne 0$ because of the invertibility of H. By repeated application of (3.4) we have along the main anti-diagonal of H the entries alternating between c and $-c$, i.e., $H_{i\ n+1-i} = (-1)^{\frac{n}{2}-i} \cdot c$. This determines all entries H_{ij} for $i+j < n+2$. Since $H_{\frac{n}{2}\ \frac{n}{2}} = 0$ it follows from (3.4) that $H_{\frac{n}{2}\ \frac{n}{2}+2} = -c$. Continuing in this way gives that for $i = 2, \ldots, \frac{n}{2}$ we have $H_{i\ n+2-i} = c \cdot q_{i\ \frac{n}{2}-i+2}$, where q_{ij} is defined as in Definition 2.3. This then defines all entries of H_{ij} for $i+j < n+3$. It is important to note that S^THS preserves the previous properties for H.

If we show that there is an invertible S such that $S^{-1}AS = A$ and the right lower corner block of S^THS is zero, then by repeated application of (3.4) the bottom row of the upper right corner block of S^THS has only entries alternating between $-c$ and c. From this point, again by repeated application of (3.4), one proves that the upper right corner block of S^THS is given by $c \cdot Q_n$. Finally take $S = \frac{1}{\sqrt{c}} I$ to finish the proof.

It remains to find such an S. We do this by changing the Jordan basis step by step. First we define a new Jordan basis as follows: let

$$z_1 = x_1, \quad z_2 = x_2, \quad z_3 = x_3 + h_1 x_1, \quad z_4 = x_4 + h_1 x_2, \ldots.$$

So, in general we have

$$z_j^{(i)} = x_j \quad \text{for} \quad j = 1, 2 \quad \text{and} \quad z_j^{(i)} = x_j + h_1 x_{j-2} \quad \text{for} \quad j > 2,$$

with $h_1 \in \mathbb{C}$. The superscript is because of the iterative process in our proof. It can easily be verified that this is indeed a Jordan chain. Using this new Jordan basis, we construct a new form for H. Note that the required S must satisfy $S^{-1}AS = A$, so nothing is lost in the Jordan form of A.

In the first iteration step for finding an appropriate S, we show how to obtain a zero for the entry $H_{\frac{n}{2}+1\ \frac{n}{2}+2}$. Put $S_1 = \mathrm{toep}(1\, h_1 \cdots 0)$, then for $H^{(1)} = S_1^T H S_1$ we have

$$(S_1^T H S_1)_{ij} = h_1^2 H_{i-1\, j-1} + h_1 H_{i-1\, j} + h_1 H_{i\, j-1} + H_{ij}.$$

Hence, for $i = \frac{n}{2} + 1$, $j = \frac{n}{2} + 2$ we have that

$$\begin{aligned} H^{(1)}_{\frac{n}{2}+1\ \frac{n}{2}+2} &= h_1^2 H_{\frac{n}{2}\ \frac{n}{2}+1} + h_1 H_{\frac{n}{2}\ \frac{n}{2}+2} + h_1 H_{\frac{n}{2}+1\ \frac{n}{2}+1} + H_{\frac{n}{2}+1\ \frac{n}{2}+2} \\ &= ch_1^2 + (-c)h_1 + H_{\frac{n}{2}+1\ \frac{n}{2}+2} \\ &= c(h_1^2 - h_1) + H_{\frac{n}{2}+1\ \frac{n}{2}+2}. \end{aligned}$$

There always is a solution in order to obtain $H^{(1)}_{\frac{n}{2}+1\ \frac{n}{2}+2} = 0$, since $h_1 \in \mathbb{C}$. Now that this entry is zero, we can use (3.4) to compute all entries in $H^{(1)}_{ij}$ for $i + j = n + 3$ in terms of c. In particular we have $H^{(1)}_{\frac{n}{2}\ \frac{n}{2}+3} = c$ from (3.4). Using this equation again and the fact that $H^{(1)}_{\frac{n}{2}+2\ \frac{n}{2}+2} = 0$ (since H is skew symmetric) gives us $H^{(1)}_{\frac{n}{2}+1\ \frac{n}{2}+3} = 0$ and $H^{(1)}_{\frac{n}{2}\ \frac{n}{2}+4} = -c$.

In the next step of the algorithm we aim to obtain a zero for the $(\frac{n}{2} + 2, \frac{n}{2} + 3)$-th entry of $H^{(1)}$. For this let $S_2 = \mathrm{toep}(1\, 0\, h_2\, 0 \cdots 0)$, then the (i, j)-th entry of $H^{(2)} = S_2^T H^{(1)} S_2$ is given by

$$H^{(2)}_{ij} = (S_2^T H^{(1)} S_2)_{ij} = h_2^2 H^{(1)}_{i-2\, j-2} + h_2 H^{(1)}_{i-2\, j} + h_2 H^{(1)}_{i\, j-2} + H^{(1)}_{ij}.$$

For $i = \frac{n}{2} + 2$ and $j = \frac{n}{2} + 3$ we have

$$\begin{aligned} H^{(2)}_{\frac{n}{2}+2\ \frac{n}{2}+3} &= h_2^2 H^{(1)}_{\frac{n}{2}\ \frac{n}{2}+1} + h_2 H^{(1)}_{\frac{n}{2}\ \frac{n}{2}+3} + h_2 H^{(1)}_{\frac{n}{2}+2\ \frac{n}{2}+1} + H^{(1)}_{\frac{n}{2}+2\ \frac{n}{2}+3} \\ &= ch_2^2 + ch_2 + h_2(0) + H^{(1)}_{\frac{n}{2}+2\ \frac{n}{2}+3} \\ &= c(h_2^2 + h_2) + H^{(1)}_{\frac{n}{2}+2\ \frac{n}{2}+3}. \end{aligned}$$

This again is solvable with $h_2 \in \mathbb{C}$ and thus we have $H^{(2)}_{\frac{n}{2}+2\ \frac{n}{2}+3} = 0$. Using (3.4) a number of times we have:

$$H^{(2)}_{\frac{n}{2}+1\ \frac{n}{2}+3} = H^{(2)}_{\frac{n}{2}+1\ \frac{n}{2}+4} = 0 \quad \text{and} \quad H^{(2)}_{\frac{n}{2}\ \frac{n}{2}+5} = c.$$

Knowing these entries, we can determine all entries in $H^{(2)}_{ij}$ for $i + j = n + 4$.

Continuing in this way by induction, we obtain a sequence of matrices S_k, $k = 1, \ldots \frac{n}{2}$ in which each iteration simplifies the form of $H^{(k)}_{ij}$ leaving all previously simplified entries unchanged. Setting $S = S_{\frac{n}{2}} S_{\frac{n}{2}-1} \ldots S_2 S_1$ we get the full lower right corner block of $S^T H S$ zero. The upper right corner block will then be the matrix $c \cdot Q_n$. Finally we can scale by setting $S = \frac{1}{\sqrt{|c|}} I$. Since $H = -H^T$, this determines the whole matrix H. $\square$

3.2. Eigenvalue -1

Here we only note that the difference with $\lambda = 1$ lies in what we called the magic wand formula. If $A = J_n(-1) \oplus J_n(-1)$ and $H = \begin{bmatrix} 0 & H_{12} \\ -H_{12}^T & 0 \end{bmatrix}$, then we see that $A^T H A = H$ implies that $J_n(-1)^T H_{12} J_n(-1) = H_{12}$ must hold. If we let the (i, j)-th entry of H_{12} be denoted as $H_{i\,j}$, then the last expression gives us our magic wand formula, i.e.,

$$H_{i\,j} - H_{i\,j+1} - H_{i+1\,j} = 0.$$

The proof is analogous to the case when $\lambda = 1$ for both n odd and n even, with the exception that P_n and Q_n now depend on $\lambda = -1$.

3.3. Eigenvalues not ± 1

We consider the case where the indecomposable block is of the form

$$A = J_n(\lambda) \oplus J_n(\frac{1}{\lambda}).$$

From the results of Section 3 in [14] we may assume that the corresponding form for the skew-symmetric matrix H is given by

$$H = \begin{bmatrix} 0 & H_{12} \\ -H_{12}^T & 0 \end{bmatrix},$$

for some invertible $n \times n$ matrix H_{12}, which does not have any additional structure. Writing out $A^T H A = H$, we see that this results in

$$J_n(\lambda)^T H_{12} J_n(\frac{1}{\lambda}) = H_{12}.$$

Now compare Section 4 in [8]. The formula above is the same as formula (38) there. As a consequence, the canonical form for H_{12} is given by Theorem 4.1 in [8] and its proof. This leads to statement (iii) in Theorem 2.6 for this case.

4. The real case

In this section we consider the real case. The case where the eigenvalues are ± 1 is the same as in the complex case, the case of a real pair of eigenvalues $\lambda, \frac{1}{\lambda}$ will also be the same as in the complex case. We need to consider only the non-real eigenvalues.

So, let $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, with $\beta \neq 0$, and set

$$J_n(\gamma) = \begin{bmatrix} \gamma & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & \gamma \end{bmatrix}.$$

4.1. Complex non-unimodular eigenvalues

Proof of Theorem 2.11 *Part* (v). First we consider non-real, non-unimodular eigenvalues, so $\alpha^2 + \beta^2 \neq 1$. By Corollary 3.5 in [14] we may restrict ourselves to the case where

$$A = J_n(\gamma) \oplus J_n(\gamma^{-1}), \qquad H = \begin{bmatrix} 0 & H_{12} \\ -H_{12}^T & 0 \end{bmatrix}$$

for some invertible $2n \times 2n$ matrix H_{12}. Then $A^T H A = H$ is equivalent to

$$J_n(\gamma)^T H_{12} J_n(\gamma^{-1}) = H_{12}.$$

This, however, is exactly the same as equation (51) in [8]. That means that the canonical form for this case can be deduced as in Theorem 5.1 in [8]. □

4.2. Complex unimodular eigenvalues

In this subsection we first prove three preliminary lemmas which are needed for the proof of Case (iv) of Theorem 2.11. Assume that $H = -H^T$ is invertible, and $J_n(\gamma)^T H J_n(\gamma) = H$, where γ is as above, but with $\alpha^2 + \beta^2 = 1$, so that $\gamma^T = \gamma^{-1}$. We can rely on quite a number of results of Section 2 in [8], compare for instance Lemma 2.6 there. We consider $A = J_n(\gamma)$ with $\beta \neq 0$ and $\alpha^2 + \beta^2 = 1$. We denote $H = \left[H_{i,j}\right]_{i,j=1}^n$, where each $H_{i,j}$ is a 2×2 matrix, and $H_{j,i} = -H_{i,j}^T$. We keep the convention that comma separated subindices indicate that we are working with 2×2 blocks. This convention was used by the authors in earlier papers [5, 8, 12] as well.

Comparing the corresponding (i,j)-th blocks of A^THA and H, yields

$$\begin{aligned}
&(J_n(\gamma)^T H J_n(\gamma))_{i,j} \\
&= \begin{cases} \gamma^T H_{1,1}\gamma, & i=1, j=1, \\ \gamma^T H_{1,j-1} + \gamma^T H_{1,j}\gamma, & i=1, j>1, \\ H_{i-1,1}\gamma + \gamma^T H_{i,1}\gamma, & i>1, j=1, \\ H_{i-1,j-1} + \gamma^T H_{i,j-1} + H_{i-1,j}\gamma + \gamma^T H_{i,j}\gamma, & i>1, j>1, \end{cases} \\
&= H_{i,j}.
\end{aligned} \tag{4.1}$$

Recall from [8] the definition of $\mathcal{E}$, that is, $\mathcal{E} = \{aI_2 + bH_0 \mid a, b \in \mathbb{R}\}$ with $H_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and also $K_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
For a 2×2 matrix $H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$ we have

$$\gamma^T H\gamma = H + (h_{11} - h_{22}) \begin{bmatrix} -\beta^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{bmatrix} + (h_{12} + h_{21}) \begin{bmatrix} -\alpha\beta & -\beta^2 \\ -\beta^2 & \alpha\beta \end{bmatrix} \tag{4.2}$$

$$= H + \beta(h_{11} - h_{22})(-\beta K_0 + \alpha K_1) + \beta(h_{12} + h_{21})(-\alpha K_0 - \beta K_1). \tag{4.3}$$

Observe, if $F \in \mathcal{E}$, then always $\gamma^T F\gamma = F$, as $\mathcal{E}$ is a commutative set of matrices and $\gamma^T\gamma = I_2$.

Lemma 4.1. *For all $i, j = 1, \ldots, n$ we have $H_{i,j} \in \mathcal{E}$.*

Proof. Note that $H_{1,1} \in \mathcal{E}$ as $H_{1,1} = -H_{1,1}^T$. Now let $i = 1$, $j > 1$, then by (4.1)

$$\gamma^T H_{1,j}\gamma - H_{1,j} = -\gamma^T H_{1,j-1}.$$

Now suppose that we have already shown that $H_{1,j-1} \in \mathcal{E}$, then also $\gamma^T H_{1,j-1} \in \mathcal{E}$, so $\gamma^T H_{1,j}\gamma - H_{1,j} \in \mathcal{E}$. From (4.2) it follows that

$$\begin{aligned} -\beta(h_{11} - h_{22}) - \alpha(h_{12} + h_{21}) &= 0, \\ \alpha(h_{11} - h_{22}) - \beta(h_{12} + h_{21}) &= 0, \end{aligned}$$

or equivalently

$$\begin{bmatrix} -\beta & -\alpha \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} h_{11} - h_{22} \\ h_{12} + h_{21} \end{bmatrix} = 0.$$

As $\alpha^2 + \beta^2 = 1$ it follows that $h_{11} - h_{22} = 0$ and $h_{12} + h_{21} = 0$, so that $H_{1,j} \in \mathcal{E}$.

By induction we see that $H_{1,j} \in \mathcal{E}$ for all $j = 1, \ldots, n$, and likewise that also $H_{i,1} \in \mathcal{E}$ for all $i = 1, \ldots, n$. Using (4.1) for $i > 1, j > 1$ we have

$$\gamma^T H_{i,j}\gamma - H_{i,j} = -(H_{i-1,j-1} + \gamma^T H_{i,j-1} + H_{i-1,j}\gamma).$$

Assuming that $H_{i,j-1}, H_{i-1,j}$ and $H_{i-1,j-1}$ are all in $\mathcal{E}$, we obtain that $\gamma^T H_{i,j}\gamma - H_{i,j} \in \mathcal{E}$, and as we have seen above this implies that $H_{i,j} \in \mathcal{E}$. By induction we have proved the lemma. $\square$

The lemma allows us to state the following important result.

Lemma 4.2. *For $i > 1$ and $j > 1$ we have*

$$H_{i-1,j-1} + \gamma^T H_{i,j-1} + H_{i-1,j}\gamma = 0. \tag{4.4}$$

Proof. Since by the previous lemma $H_{i,j} \in \mathcal{E}$ for all i, j, we have that $\gamma^T H_{i,j}\gamma - H_{i,j} = 0$ for all i, j. Then (4.1) becomes (4.4). □

Next, we show that H has a (block) triangular form.

Lemma 4.3. *For $i + j \leq n$ we have $H_{i,j} = 0$.*

Proof. Since $H_{1,j} \in \mathcal{E}$ we have from (4.1) that $\gamma^T H_{1,j-1} = 0$, and by invertibility of γ it follows that $H_{1,j-1} = 0$ for $j = 2, \ldots, n$. Likewise $H_{i-1,1} = 0$ for $i = 2, \ldots, n$.

Now consider $i > 1$ and $j > 1$, and suppose that we have already shown that $H_{i-1,j-1} = 0$. By (4.4) we have $0 = \gamma^T H_{i,j-1} + H_{i-1,j}\gamma$, which implies

$$H_{i,j-1} = -H_{i-1,j}\gamma^2, \tag{4.5}$$

using again the commutativity of $\mathcal{E}$. Since we know that $H_{1,j} = 0$ for $j = 1, \ldots, n-1$, repeated application of (4.5) gives $H_{i,j} = 0$ for $i + j \leq n$. □

From here on we have to distinguish between n even and n odd.

4.2.1. The case when n is odd.

Proof of Theorem 2.11*, Part* (iv)*, Case* 1 *with n odd.* If n is odd, then $H_{\frac{n+1}{2},\frac{n+1}{2}}$ is skew symmetric, so $H_{\frac{n+1}{2},\frac{n+1}{2}} = cH_0$ for some real number c. Then by (4.5) we have $H_{\frac{n-1}{2},\frac{n+3}{2}} = -cH_0(\gamma^T)^2$, and $H_{\frac{n-3}{2},\frac{n+5}{2}} = cH_0(\gamma^T)^4$, and so on.

Up to this point we have derived results that hold for any Jordan basis. The next step is to construct a special Jordan basis such that H is in the canonical form. Equivalently, we find a matrix S such that $S^{-1}J_n(\gamma)S = J_n(\gamma)$, while $S^T HS$ is in the canonical form. This will be done step by step. Ultimately we shall show that it is possible to take S such that $H_{i,j} = 0$ for $i, j > \frac{n+1}{2}$. In addition, the matrix S is such that the block entry $H_{\frac{n+1}{2},\frac{n+3}{2}}$ is equal to $\frac{1}{2}c\gamma^T H_0$. Note that the latter is in accordance with (4.4).

We follow a similar line of reasoning as in the proof in Section 2.3 in [8]. For an invertible block upper triangular Toeplitz matrix S with block entries in $\mathcal{E}$ we have that $S^{-1}J_n(\gamma)S = J_n(\gamma)$. We shall use such matrices to construct a canonical form via $S^T HS$.

First, let $S_1 = \mathrm{toep}(h_1, 0, \cdots, 0)$, with $0 \neq h_1 \in \mathcal{E}$. Let $H^{(1)} = S_1^T HS_1$. Then $(S_1^T HS_1)_{i,j} = h_1^T H_{i,j}h_1 = h_1^T h_1 H_{i,j} = dH_{i,j}$ for some positive real number d. So we see that we can scale the entry in the $(\frac{n+1}{2}, \frac{n+1}{2})$-position to εH_0 with $\varepsilon = \pm 1$. After that we can extract ε from every entry in $H^{(1)}$, and put it in front of the matrix. This way we may assume that $H_{\frac{n+1}{2},\frac{n+1}{2}} = H_0$. From now on we shall assume that this is the case.

Next, consider the entry at position $(\frac{n+1}{2}, \frac{n+3}{2})$. By (4.4) we have

$$H_0 + \gamma^T H^{(1)}_{\frac{n+3}{2},\frac{n+1}{2}} + H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}}\gamma = 0.$$

Since $H^{(1)}_{\frac{n+3}{2},\frac{n+1}{2}} = -\left(H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}}\right)^T$, this amounts to

$$H_0 - \gamma^T\left(H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}}\right)^T + H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}}\gamma = 0.$$

One easily checks that since $H^{(1)}_{\frac{n+1}{2}\ \frac{n+3}{2}}$ is in $\mathcal{E}$, that for some real number d

$$H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}} = -\frac{1}{2}\gamma^T H_0 + d\gamma^T. \tag{4.6}$$

Let $S_2 = \mathrm{toep}(I_2, h_2, 0, \cdots, 0)$ with $0 \neq h_2 \in \mathcal{E}$ and put $H^{(2)} = S_2^T H^{(1)} S_2$. Then

$$H^{(2)}_{i,j} = H^{(1)}_{i,j} + H^{(1)}_{i,j-1}h_2 + h_2^T H^{(1)}_{i-1,j} + h_2^T H^{(1)}_{i-1,j-1}h_2.$$

For $i+j \le n+1$ we have $H^{(2)}_{i,j} = H^{(1)}_{i,j}$. Indeed, in this case by Lemma 4.3, $H^{(1)}_{i,j-1} = H^{(1)}_{i-1,j} = H^{(1)}_{i-1,j-1} = 0$. For $i = \frac{n+1}{2}, j = \frac{n+3}{2}$ we have from Lemma 4.2 and 4.3 that

$$\begin{aligned}
H^{(2)}_{\frac{n+1}{2},\frac{n+3}{2}} &= H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}} + H^{(1)}_{\frac{n+1}{2},\frac{n+1}{2}}h_2 + h_2^T H^{(1)}_{\frac{n-1}{2},\frac{n+3}{2}} \\
&= H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}} + H_0 h_2 - h_2^T H_0(\gamma^T)^2 \\
&= H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}} + \gamma^T(H_0(\gamma h_2 - h_2^T\gamma^T)),
\end{aligned}$$

where in the last equality we used the commutativity of $\mathcal{E}$ and $\gamma^T\gamma = I_2$. Now observe that $H_0(\gamma h_2 - h_2^T\gamma^T)$ is a scalar multiple of the identity matrix depending on $h_2 \in \mathcal{E}$. This implies that by choosing h_2 appropriately we can obtain using (4.6)

$$H^{(2)}_{\frac{n+1}{2},\frac{n+3}{2}} = -\frac{1}{2}\gamma^T H_0.$$

In the next steps we shall show that we can choose the Jordan basis so that the entries $H_{i,j}$ with both $i > \frac{n+1}{2}$ and $j > \frac{n+1}{2}$ are all zero. For this it suffices, by (4.4), to show that S can be chosen so that $H_{i,i} = 0$ and $H_{i,i+1} = 0$ for $i > \frac{n+1}{2}$.

As a first step, take $S_3 = \mathrm{toep}(I_2, 0, h_3, 0\cdots, 0)$ with $h_3 \in \mathcal{E}$, and let $H^{(3)} = S_3^T H^{(2)} S_3$. Then $H^{(3)}_{i,j} = H^{(2)}_{i,j}$ for $i+j \le n+2$, as one easily checks, and

$$\begin{aligned}
H^{(3)}_{\frac{n+3}{2},\frac{n+3}{2}} &= H^{(2)}_{\frac{n+3}{2},\frac{n+3}{2}} + h_3^T H^{(2)}_{\frac{n-1}{2},\frac{n+3}{2}} + H^{(2)}_{\frac{n+3}{2},\frac{n-1}{2}} + h_3^T H^{(2)}_{\frac{n-1}{2},\frac{n-1}{2}}h_3 \\
&= H^{(2)}_{\frac{n+3}{2},\frac{n+3}{2}} - h_3^T H_0(\gamma^T)^2 - \gamma^2 H_0 h_3.
\end{aligned}$$

Now $H^{(2)}_{\frac{n+3}{2},\frac{n+3}{2}}$ and $H^{(3)}_{\frac{n+3}{2},\frac{n+3}{2}}$ are both skew symmetric, hence a real multiple of H_0. Since $h_3^T(\gamma^T)^2 + \gamma^2 h_3$ is a multiple of the identity it is clear that we can choose h_3 such that $H^{(3)}_{\frac{n+3}{2},\frac{n+3}{2}} = 0$.

By (4.4), and the fact that $H^{(3)}_{\frac{n+5}{2},\frac{n+3}{2}} = -(H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}})^T$ we obtain that

$$-\gamma^T(H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}})^T + H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}}\gamma = 0.$$

In turn, this implies that $H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}}$ is a real multiple of γ^T.

As a second step, take $S_4 = \mathrm{toep}(I_2, 0, 0, h_4, 0 \cdots, 0)$ with $h_4 \in \mathcal{E}$, and let $H^{(4)} = S_4^T H^{(3)} S_4$. Then $H^{(4)}_{i,j} = H^{(3)}_{i,j}$ for $i + j \leq n + 3$, as one easily checks, and

$$\begin{aligned} H^{(4)}_{\frac{n+3}{2},\frac{n+5}{2}} &= H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}} + h_4^T H^{(3)}_{\frac{n-3}{2},\frac{n+5}{2}} + H^{(3)}_{\frac{n+3}{2},\frac{n-1}{2}} h_4 + h_4^T H^{(3)}_{\frac{n-3}{2},\frac{n-1}{2}} h_4 \\ &= H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}} + h_4^T H^{(3)}_{\frac{n-3}{2},\frac{n+5}{2}} + H^{(3)}_{\frac{n+3}{2},\frac{n-1}{2}} h_4. \end{aligned}$$

Since $H^{(3)}_{\frac{n-3}{2},\frac{n+5}{2}} = H_0(\gamma^T)^4$ and $H^{(3)}_{\frac{n+3}{2},\frac{n-1}{2}} = -H_0\gamma^2$ (as noted in the beginning of the proof) the latter equation becomes

$$\begin{aligned} H^{(4)}_{\frac{n+3}{2},\frac{n+5}{2}} &= H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}} + h_4^T H_0(\gamma^T)^4 - H_0\gamma^2 h_4 \\ &= H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}} + \gamma^T H_0(h_4^T(\gamma^T)^3 - \gamma^3 h_4). \end{aligned}$$

Now $H_0(h_4^T(\gamma^T)^3 - \gamma^3 h_4)$ is a real multiple of I_2 and as we already know that $H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}}$ is a real multiple of γ^T, it follows that we can take h_4 so that $H^{(4)}_{\frac{n+3}{2},\frac{n+5}{2}} = 0$.

We can now continue in this manner by an induction argument (compare [8], Section 2) and by setting $S = S_{\frac{n+1}{2}} \cdots S_2\, S_1$. □

4.2.2. The case when n is even.

Proof of Theorem 2.11*, Part* (iv)*, Case* 2 *with n even.* If n is even, then by (4.5) $H_{\frac{n}{2}+1,\frac{n}{2}} = -H_{\frac{n}{2},\frac{n}{2}+1}\gamma^2$. Since also $H_{\frac{n}{2}+1,\frac{n}{2}} = -H^T_{\frac{n}{2},\frac{n}{2}+1}$ we have $H^T_{\frac{n}{2},\frac{n}{2}+1} = H_{\frac{n}{2},\frac{n}{2}+1}\gamma^2$. A straightforward computation then gives $H_{\frac{n}{2},\frac{n}{2}+1} = c\gamma^T$ for some real number c. Then $H_{\frac{n}{2}-1,\frac{n}{2}+2} = -c(\gamma^T)^3$, $H_{\frac{n}{2}-2,\frac{n}{2}+3} = c(\gamma^T)^5$, and so on.

Up to this stage we have derived a "form" that holds for a general Jordan basis. The next step is to construct a special Jordan basis such that the "form" for H is transformed into the canonical form given in Theorem 2.11. Equivalently, we find step by step a matrix S such that $S^{-1}J_n(\gamma)S = J_n(\gamma)$, while S^THS is in the canonical form. We shall show that it is possible to take S such that $H_{i,j} = 0$ for $i, j > \frac{n}{2}$.

First we take $S_1 = \mathrm{toep}(h_1, 0, \cdots, 0)$, with $0 \neq h_1 \in \mathcal{E}$. Then $H^{(1)} = S_1^T H S_1$ has block entries equal to $h_1^T h_1$ times the corresponding entry in H. Since $h_1^T h_1$ is a positive multiple of the identity this means we can arrange for $H^{(1)}_{\frac{n}{2},\frac{n}{2}+1} = \varepsilon\gamma^T$, with $\varepsilon = \pm 1$. Taking out ε from all the entries, we may assume that $H^{(1)}_{\frac{n}{2},\frac{n}{2}+1} = \gamma^T$, and this will be assumed from now on.

Take $S_2 = \mathrm{toep}(I_2, h_2, 0 \cdots, 0)$ with $h_2 \in \mathcal{E}$, and put $H^{(2)} = S_2^T H^{(1)} S_2$. Then

$$\begin{aligned} H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+1} &= H^{(1)}_{\frac{n}{2}+1,\frac{n}{2}+1} + h_2^T H^{(1)}_{\frac{n}{2},\frac{n}{2}+1} + H^{(1)}_{\frac{n}{2}+1,\frac{n}{2}} h_2 + h_2^T H^{(1)}_{\frac{n}{2},\frac{n}{2}} h_2 \\ &= H^{(1)}_{\frac{n}{2}+1,\frac{n}{2}+1} + h_2^T \gamma^T - \gamma h_2. \end{aligned}$$

Since $h_2^T\gamma^T - \gamma h_2$ is a multiple of H_0, and since $H^{(1)}_{\frac{n}{2}+1,\frac{n}{2}+1}$ is a skew-symmetric 2×2 matrix, and hence also is a multiple of H_0, it is possible to choose h_2 so that $H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+1} = 0$.

Next, take $S_3 = \mathrm{toep}(I_2, 0, h_3, 0, \cdots, 0)$ and put $H^{(3)} = S_3^T H^{(2)} S_3$. Then

$$\begin{aligned} H^{(3)}_{\frac{n}{2}+1,\frac{n}{2}+2} &= H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2} + h_3^T H^{(2)}_{\frac{n}{2}-1,\frac{n}{2}+2} + H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}} h_3 + h_3^T H^{(2)}_{\frac{n}{2},\frac{n}{2}} h_3 \\ &= H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2} - h_3^T(\gamma^T)^3 - \gamma h_3 \\ &= H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2} - \gamma^T(h_3^T(\gamma^T)^2 + \gamma^2 h_3). \end{aligned}$$

Now by (4.4) we have

$$0 = H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2}\gamma + \gamma^T H^{(2)}_{\frac{n}{2}+2,\frac{n}{2}+1},$$

and since $H^{(2)}_{\frac{n}{2}+2,\frac{n}{2}+1} = -(H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2})^T$ this amounts to

$$0 = H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2}\gamma - \gamma^T (H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2})^T.$$

This implies that $H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2} = d\gamma^T$ for some real number d. Then it follows that it is possible to choose h_3 such that $H^{(3)}_{\frac{n}{2}+1,\frac{n}{2}+2} = 0$.

We can now continue and finish the argument by induction showing that $H_{i,j} = 0$ for $i, j > \frac{n}{2}$. This in turn implies by repeated application of (4.4) that the entries in the right upper corner block of H have the form as described in the theorem by setting $S = S_1 S_2 \dots S_{\frac{n}{2}}$. □

5. The complex H-symplectic case

To complete the paper, let us consider finally the case where A and H are complex matrices, and, with $H = -H^*$ invertible and $A^*HA = H$. As we observed in the introduction, the matrix A is then iH-unitary, and iH is Hermitian. Thus we can apply the result of [8], Theorem 6.1, to arrive at the following theorem.

Theorem 5.1. *Let H be a complex skew-Hermitian invertible matrix, and let A be complex H-symplectic, so $A^*HA = H$. Then the pair (A, H) can be decomposed as*

$$S^{-1}AS = \bigoplus_{l=1}^{p} A_l, \quad S^*HS = \bigoplus_{l=1}^{p} H_l,$$

where the pairs (A_l, H_l) have one of the following forms with n depending on l

(i) $\sigma(A_l) = \{\lambda, \bar{\lambda}^{-1}\}$ *with* $|\lambda| \neq 1$, *and*

$$A_l = J_n(\lambda) \oplus J_n(\bar{\lambda}^{-1}), \quad H_l = \begin{bmatrix} 0 & H_{12} \\ -H_{12}^* & 0 \end{bmatrix},$$

where H_{12} *has one of the following two forms depending on whether* n *is odd or even:*

Case 1: $H_{12} = -i \begin{bmatrix} Z_n & P_n(\bar{\lambda}) \\ P_n(\bar{\lambda}^{-1})^T & 0 \end{bmatrix}$ *when* n *is odd,*

Case 2: $H_{12} = -i \begin{bmatrix} 0 & Q_n(\bar{\lambda}) \\ -\bar{\lambda}^{-2} Q_n(\bar{\lambda}^{-1})^T & 0 \end{bmatrix}$ *when* n *is even.*

(ii) $\sigma(A_l) = \{\lambda\}$ *with* $|\lambda| = 1$, *and the pair* (A_l, H_l) *has one of the following two forms:*

Case 1: $(J_n(\lambda), -i\varepsilon \begin{bmatrix} Z_n & P_n(\bar{\lambda}) \\ P_n(\lambda)^T & 0 \end{bmatrix})$ *with* $\varepsilon = \pm 1$ *and* n *is odd,*

Case 2: $(J_n(\lambda), \varepsilon \begin{bmatrix} 0 & \bar{\lambda} Q_n(\bar{\lambda}) \\ -\lambda Q_n(\lambda)^T & 0 \end{bmatrix})$ *with* $\varepsilon = \pm 1$ *and* n *is even.*

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G.J. Groenewald and D.B. Janse van Rensburg
Department of Mathematics, Unit for BMI, North-West University
Potchefstroom, 2531, South Africa
e-mail: Gilbert.Groenewald@nwu.ac.za
Dawie.JanseVanRensburg@nwu.ac.za

A.C.M. Ran
Department of Mathematics, FEW, VU university Amsterdam
De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
and
Unit for BMI, North-West University
Potchefstroom, South Africa
e-mail: a.c.m.ran@vu.nl